TECHNISCHE UNIVERSITÄT DRESDEN

Communications Theory

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Chair of Communications Theory

STOCHASTIC SIGNALS AND SYSTEMS

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ABBREVIATIONS

ACF	Autocorrelation function
CCF	Cross-correlation function
CDF	Cumulative distribution function
PDF	Probability density function

EXERCISES

8 STOCHASTIC SIGNALS

8.1. The illumination of a room is supplied by two serially connected lamps L_1 and L_2, independently failing with the probabilities p_1 and p_2. What is the probability for the failure of the room's illumination?

8.2. An electric circuit (see Figure 8.2) contains four ohmic resistors, independently failing with the probabilities p_1, p_2, p_3, and p_4 (i. e. $R_i \to \infty$, $i \in \{1, 2, 3, 4\}$). Calculate the probability for the discontinuity of the current I!

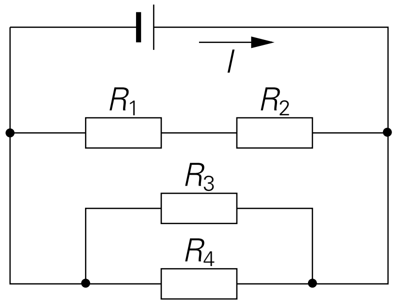


Figure 8.2

8.3. Two shooters fire at a target. The probability to hit the target is 0.8 for the first shooter and 0.9 for the second one. What is the probability that the target will be hit?

8.4. Numerically coded control commands of the type 111 and 000 are transmitted via a disturbed channel. The transmission probability for the first type is 0.7 and for the second one is 0.3. Each token (0 or 1) is transferred correctly with a probability of 0.8.

a) What is the probability for the reception of the control command 101?

b) If the received code is 101, what is the probability that

   α) 111, β) 000

   was transmitted?

8.5. A random variable X is given with the cumulative distribution function (CDF) F_X:

$$F_X(\xi) = \begin{cases} 0 & \xi \leq -1, \\ 1 - \xi^2 & -1 < \xi \leq 0, \\ 1 & \xi > 0. \end{cases}$$

a) Calculate the probability density function (PDF) f_X and draw a sketch!

b) What is the probability for X having a value smaller than $-\frac{1}{2}$?

c) Calculate the probability $P\left\{-\frac{1}{3} \leq X < 2\right\}$ using

   α) the cumulative distribution function (CDF),
   β) the probability density function (PDF)!

8.6. Calculate the expected value (mean) $\mathrm{E}(X)$, the quadratic mean $\mathrm{E}(X^2)$, and the variance $\mathrm{Var}(X)$

a) of a discrete random variable X with corresponding CDF F_X according to Figure 8.6a,

b) of a continuous random variable X with corresponding CDF F_X

$$F_X(\xi) = \begin{cases} 0 & \xi \leq 0, \\ \xi^2 & 0 < \xi \leq 1, \\ 1 & \xi > 1 \end{cases}$$

according to Figure 8.6b!

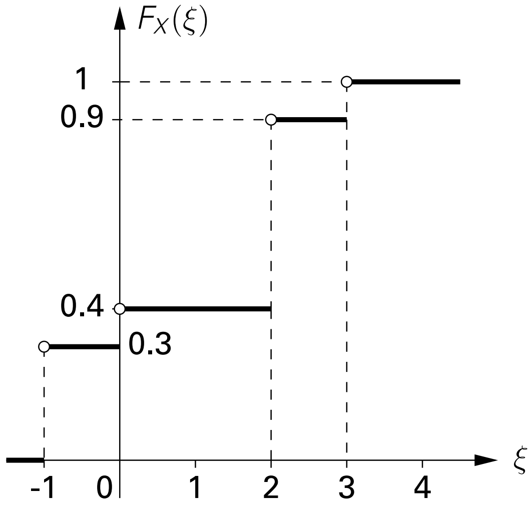


Figure 8.6a

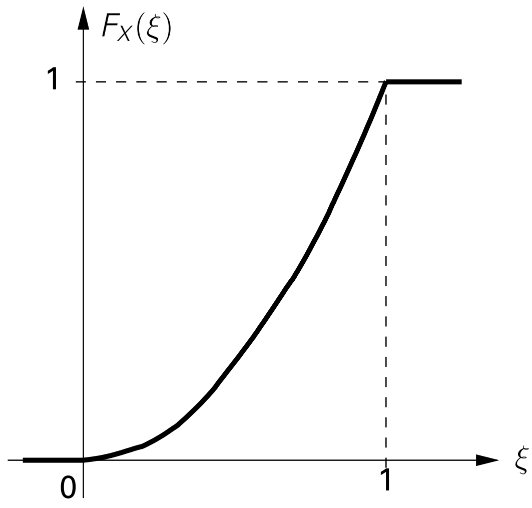


Figure 8.6b

8.7. The life cycle (the time from start-up to the failure) of an electronic system is given by the random variable X with the PDF f_X:

$$f_X(x) = \begin{cases} a\,\mathrm{e}^{-ax} & x \geq 0 \quad (a > 0) \\ 0 & x < 0. \end{cases}$$

The average lifetime is 10 years. Calculate the probability that

a) the system operates reliably for at least 3 years,

b) the system will operate reliably for another 2 years, if it is known that the system has been working for 3 years already!

8.8. The life cycle of a component can be approximated by the random variable X with the PDF f_X:

$$f_X(x) = \begin{cases} \lambda^2 x\,\mathrm{e}^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad (\lambda = 0.25/\,\text{year}).$$

a) Calculate the probability that the component does not fail within 6 years!

b) A device consists of 4 of these components, failing independently of each other. Calculate the probability that the device does not fail within 6 years!

8.9. A collection of electronic devices contains rejects of 5% (faulty devices). At least how many devices should a random sample contain (i. e. how many devices have to be checked) in order to find at least one faulty device with a probability of not less than 0.9?

8.10. A random vector $X = (X_1, X_2)$ is uniformly distributed in a rectangle B_1 (Figure 8.10), i. e., the PDF is

$$f_X(x_1, x_2) = \begin{cases} \frac{1}{ab} & (x_1, x_2) \in B_1 \\ 0 & (x_1, x_2) \notin B_1 \end{cases} \quad (a > b > 0).$$

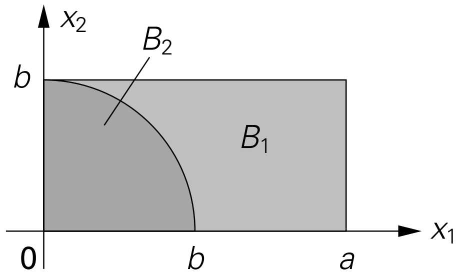


Figure 8.10

a) What is the probability for X to lie within the quarter circle area B_2?

b) Calculate the probability for X_1 having a value greater than b (X_2 arbitrary)!

8.11. Given a random vector $X = (X_1, X_2)$ with the CDF F_X.

Calculate as a function of F_X

a) $P\{X \in B_1\}$,

b) $P\{X \in B_1 \,|\, X \in B_2\}$!

(See Figure 8.11.)

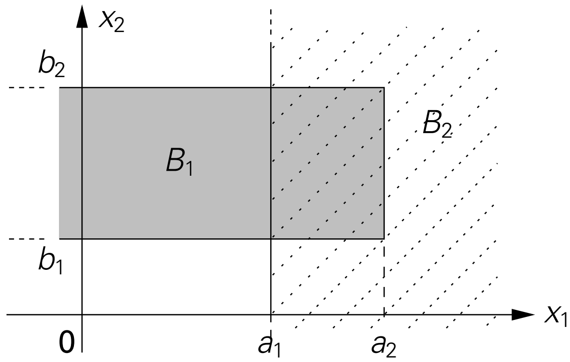


Figure 8.11

8.12. During a message transmission, 1% of all characters are faultily received. What is the probability that in a text of 200 characters there is

a) no

b) at most one

faulty character.

8.13. A random vector $X = (X_1, X_2)$ is uniformly distributed in a rectangle B, i. e. the PDF $f_X(x_1, x_2)$ is constant for $(x_1, x_2) \in B$. (See Figure 8.13.)

a) What is the PDF f_X?

b) Calculate the marginal PDFs f_{X_1} and f_{X_2} and draw a sketch for each!

c) Calculate $P\{X_1 \geq 1\}$!

d) Calculate the probability for X_2 having a value greater than X_1!

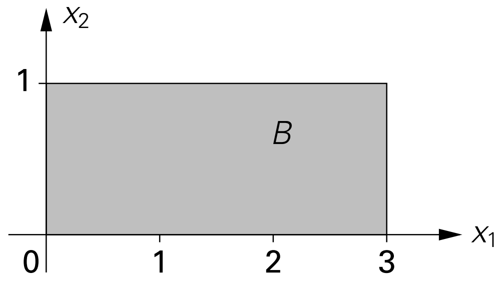


Figure 8.13

8.14. A random vector $X = (X_1, X_2)$ is distributed in a rectangle B with the PDF f_X:

$$f_X(x_1, x_2) = \begin{cases} \frac{x_1}{\pi} & (x_1, x_2) \in B, \\ 0 & (x_1, x_2) \notin B. \end{cases}$$

a) Calculate $f_{X_1}(x_1 \mid x_2)$ and $f_{X_2}(x_2 \mid x_1)$. Find out whether the components X_1 and X_2 of X are stochastically independent of each other!

b) Calculate the probability for X_1 having a value smaller than 0.5?

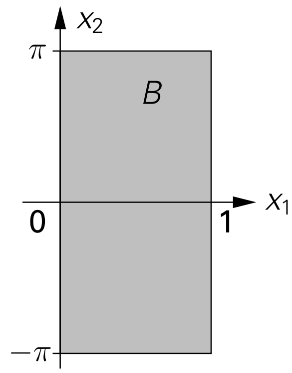


Figure 8.14

8.15. A random vector $X = (X_1, X_2, X_3)$ is uniformly distributed inside the sphere ${x_1}^2 + {x_2}^2 + {x_3}^2 \leq R^2$, i. e., the PDF of X is constant inside the sphere. What is this PDF?

8.16. Given are three stochastically independent random variables X_1, X_2, and X_3 with $\mathrm{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = {\sigma_i}^2 \quad (i \in \{1, 2, 3\})$.

a) Calculate the variance $\mathrm{Var}(Y)$ of $Y = a_1X_1 + a_2X_2 + a_3X_3 \quad (a_i \in \mathbb{R})$!

b) Determine specifically the variances $\mathrm{Var}(X_1 + X_2)$ and $\mathrm{Var}(X_1 - X_2)$!

8.17. Consider a random process $X = (X_t)_{t \in T}$ with $X_t = X(t) = X_1 \sin(\omega_0 t - X_2)$, wherein X_1 and X_2 are random variables uniformly distributed in the interval $(0, 2\pi]$. Specify some realisations of X and draw their curves!

8.18. A noise voltage across an ohmic resistor R can be approximated by a stationary random process X with the PDF f_X:

$$f_X(x, t) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right) \qquad (a > 0)$$

Calculate (for a fixed time t)

a) the probability, that the voltage exceeds a given value $a_0 > 0$;

b) the expected value of the voltage;

c) the expected value of the power at R!

d) What is the result using $a = 1$ V, $a_0 = 2$ V, and $R = 3$ ˙?

Note to b): $\displaystyle\int x\,\mathrm{e}^{cx}\,\mathrm{d}x = \frac{\mathrm{e}^{cx}}{c^2}(cx-1) + C$

Note to c): $\displaystyle\int x^2\,\mathrm{e}^{cx}\,\mathrm{d}x = \frac{\mathrm{e}^{cx}}{c^3}(x^2c^2 - 2cx + 2) + C$

8.19. The circuit given in Figure 8.19 contains a noise voltage source and a noise current source. The current flowing through R_2 can be represented by the random process

$$I_2 = \frac{U - IR_1}{R_1 + R_2}$$

U and I are stationary (and jointly stationary) random processes with the correlation functions s_U, s_I, and s_{UI}.

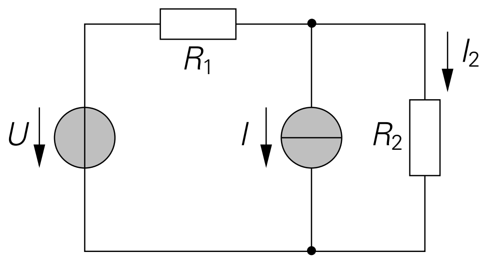


Figure 8.19

a) Calculate $s_{I_2}(\tau)$ as a function of $s_U(\tau)$, $s_I(\tau)$, and $s_{UI}(\tau)$!

b) What is the mean value of the power at the resistor R_2?

8.20. Given the random process Y:

$$Y(t) = X_1 \cos\omega_0 t + X_2 \sin\omega_0 t \qquad (\omega_0 \in \mathbb{R},\ \text{constant}),$$

wherein X_1 and X_2 are independent random variables with

$$\mathrm{E}(X_1) = \mathrm{E}(X_2) = 0 \text{ and } \mathrm{E}({X_1}^2) = \mathrm{E}({X_2}^2) = \sigma^2.$$

a) Calculate the expected value $\mathrm{E}(Y(t)) = m_Y(t)$!

b) Calculate the autocorrelation function $\mathrm{E}(Y(t_1)Y(t_2)) = s_Y(t_1, t_2)$!

c) Is the process Y wide-sense stationary?

8.21. Let s_X be the autocorrelation function of a stationary random process $X = (X_t)_{t\in T}$. Show that

$$|\, s_X(\tau)\, | \le s_X(0) \qquad (\tau = t_2 - t_1;\ t_1, t_2 \in T) \text{ holds.}$$

Note:
Calculate the (nonnegative) expression $\mathrm{E}\left((X(t) \pm X(t+\tau))^2\right) \ge 0$!

8.22. A noise voltage across an ohmic resistor R can be approximated by a stationary random process U with a zero mean and the power density spectrum S_U:

$$S_U(\omega) = \begin{cases} S_0 & -\omega_0 \le \omega \le +\omega_0 \\ 0 & \omega < -\omega_0,\ \omega > +\omega_0 \end{cases} \qquad (S_0 > 0,\ \text{constant}).$$

a) What is the power density spectrum and the autocorrelation function of the current I through the ohmic resistor R!

b) Calculate the mean power input of R!

8.23. The current through an ohmic resistor R can be approximated by a stationary Gaussian process X, where

$$m_X(t) = 0 \ \text{ and } \ s_X(\tau) = A^2\, \mathrm{e}^{-\alpha|\tau|} \qquad (A, \alpha \in \mathbb{R},\ \alpha > 0).$$

a) Calculate the power density spectrum $S_X(\omega)$!

b) Calculate the mean power input of R!

c) What is the PDF $f_X(x, t)$?

d) What is the PDF $f_X(x_1, t_1; x_2, t_2)$? $\quad (\tau = t_2 - t_1)$

8.24. Let X be a stationary Gaussian process with a zero mean and the autocorrelation function s_X:

$$s_X(\tau) = A^2\, \mathrm{e}^{-\alpha|\tau|} \left(\cos \beta\tau - \frac{\alpha}{\beta} \sin \beta|\tau| \right) \qquad (A > 0,\ \alpha > 0,\ \beta > 0).$$

Calculate the probability for $X(t)$ having a value greater than b!
Numerical example: $A = 1$ V, $\alpha = 10^4\ \mathrm{s}^{-1}$, $\beta = 10^5\ \mathrm{s}^{-1}$, $b = 0.5$ V.
Note: Gauss error function

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_0^u \exp\left(-\frac{v^2}{2} \right) \mathrm{d}v;$$

$$\Phi(u) = -\Phi(-u); \quad \Phi(\infty) = 0.5; \quad \Phi(0.5) \approx 0.1915$$

9 STATIC SYSTEMS

9.1. A non-linear static system (see Figure 9.1a) is given with an exponential characteristic curve $\varphi : \mathbb{R} \to \mathbb{R}$, $y = \varphi(x) = \mathrm{e}^{3x}$.

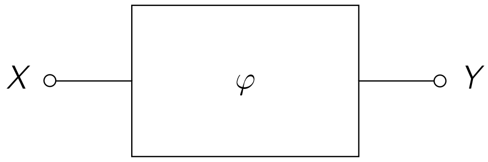


Figure 9.1a

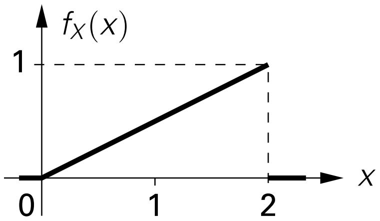


Figure 9.1b

The input values of this system can be approximated by a random variable X with a triangular distribution (the corresponding PDF is shown in Figure 9.1b). Calculate the PDF of the random variable Y at the output of the system! Draw the curve of the PDF $f_Y(y)$!

9.2. A static system is shown in Figure 9.2. The input values of this system are given by the random vector $X = (X_1, X_2)$ with the PDF f_X.
Calculate the PDF f_Y of the random vector $Y = (Y_1, Y_2)$ at the output of the system

a) generally for any desired f_X,

b) specifically for

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{{x_1}^2 + {x_2}^2}{2\sigma^2}\right) \quad (\sigma > 0)!$$

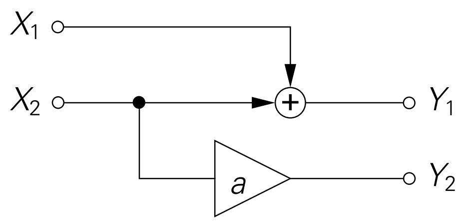


Figure 9.2

9.3. The RANDOM function of a computer generates pseudo-random numbers which can approximately be characterised as a uniformly distributed random variable in the interval $(0, 1)$.
Which arithmetic operation has to be applied to these numbers in order to obtain random numbers with a Cauchy distribution with the PDF f_Y:

$$f_Y(y) = \frac{1}{\pi} \frac{1}{y^2 + 1}? \quad \text{(see Figure 9.3)}$$

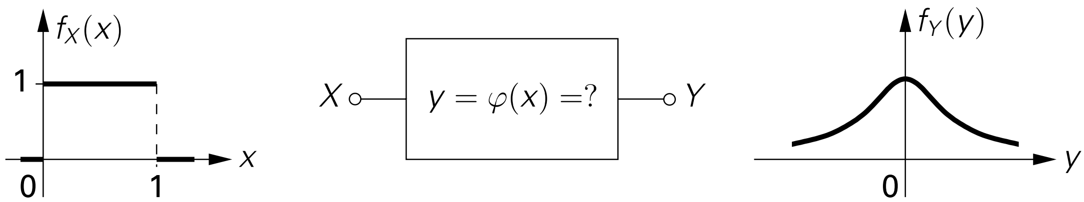


Figure 9.3

9.4. A static system is shown in Figure 9.4. X_1 and X_2 are stochastically independent random variables for which $\mathrm{E}(X_1) = \mathrm{E}(X_2) = 0$ and $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \sigma^2$ hold.

a) Calculate $\mathrm{E}(Y_1)$ and $\mathrm{E}(Y_2)$!

b) Calculate $\mathrm{Var}(Y_1)$ and $\mathrm{Var}(Y_2)$!

c) Calculate $\mathrm{Cov}(Y_1, Y_2)$!

d) What is the correlation coefficient $\varrho(Y_1, Y_2)$?

e) Calculate $f_Y(y_1, y_2)$, if

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{{x_1}^2 + {x_2}^2}{2\sigma^2}\right)!$$

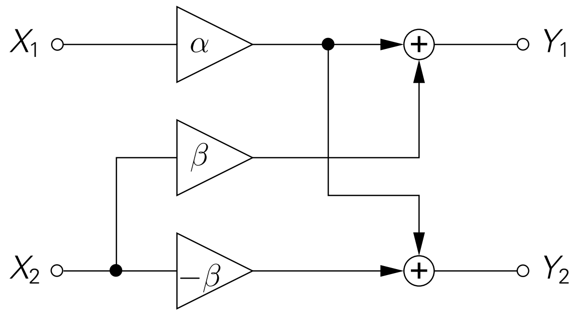


Figure 9.4

9.5. The input of a rectifier shown in Figure 9.5 with the characteristic curve φ:

$$y = \varphi(x) = \begin{cases} \mathrm{e}^{ax} - 1 & x \geq 0, \\ 0 & x < 0 \end{cases}$$

can be approximated by a non-stationary random process X with the PDF f_X, with $f_X(x, t) = 0$ for all $t \in T$ if $x < 0$.

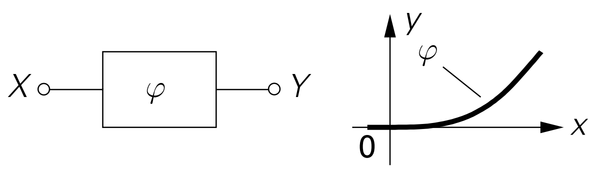


Figure 9.5

a) Calculate $f_Y(y, t)$ in general!

b) What is the result specifically for

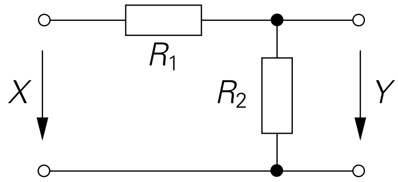
$$f_X(x, t) = \begin{cases} \dfrac{\alpha}{1 + \beta^2 t^2} \exp\left(\dfrac{-\alpha x}{1 + \beta^2 t^2}\right) & x \geq 0, \\ 0 & x < 0? \end{cases}$$

For the constants $a > 0$, $\alpha > 0$, $\beta > 0$ hold.

9.6. The autocorrelation function s_X and the PDF f_X of the stationary random process X shown in Figure 9.6 are given as:

$$s_X(\tau) = 2A^2\,\mathrm{e}^{-\alpha|\tau|} \cos\beta\tau$$

$$f_X(x, t) = \frac{1}{2A} \exp\left(-\frac{|x|}{A}\right) \quad (A > 0,\ \alpha > 0,\ \beta > 0).$$



$(X \Leftrightarrow U_1,\ Y \Leftrightarrow U_2)$

Figure 9.6

a) Calculate the autocorrelation function of the process Y!

b) Calculate the one-dimensional PDF of the process Y!

c) For an arbitrary time t and ($a > 0$), calculate the probability for $Y(t) > a$.

d) What is the result of c), if $A = 1$ V, $a = 2$ V, $R_1 = 1$ ˙, and $R_2 = 2$ ˙ are given?

9.7. The circuit shown in Figure 9.7 contains two adders and two ideal amplifiers with the amplification factors v_1 and v_2. The processes X (input process), U, and V (disturbance processes) are stationary and independent random processes with the mean values $m_X = m_U = m_V = 0$.

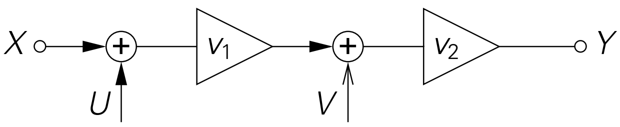


Figure 9.7

The power density spectrum of the process X is given by

$$S_X(\omega) = \frac{A^2}{\omega^2 + a^2} \qquad (A > 0,\ a > 0)$$

whereas U and V are white noise processes with

$$S_U(\omega) = S_V(\omega) = S_0 \qquad (S_0 > 0).$$

a) Calculate the cross-correlation function of the processes X and Y!

b) Calculate the power density spectrum of the process Y!

9.8. A noise voltage across a diode with the current voltage characteristic

$$i = \varphi(u) = I_0 \left(\exp\left(\frac{u}{U_0}\right) - 1 \right) \qquad (I_0 > 0,\ U_0 > 0)$$

can be approximated by a stationary random process U with the PDF f_U:

$$f_U(u, t) = \begin{cases} \dfrac{1}{U_0} & 0 \le u \le U_0 \\ 0 & u < 0,\ u > U_0 \end{cases} .$$

a) Calculate the PDF f_I of the current I and draw the curve of $f_I(i, t)$!

b) Calculate the mean of the current I by using the equation

$$\mathrm{E}(\varphi(X)) = \int_{-\infty}^{\infty} \varphi(x) f_X(x)\,\mathrm{d}x!$$

10 DYNAMIC SYSTEMS

10.1. Show that a stationary random process X with the autocorrelation function s_X is mean-square continuous, if and only if $s_X(\tau)$ is continuous in $\tau = 0$.
Note: Examine the equation

$$||X(t+\tau) - X(t)||^2 = \mathrm{E}\left((X(t+\tau) - X(t))^2\right) \quad \text{for} \quad \tau \to 0!$$

10.2. Let X be a mean-square differentiable random process with mean m_X and autocorrelation function s_X. Show that,

a) $m_{\dot{X}}(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} m_X(t)$,

b) $s_{\dot{X}X}(t_1, t_2) = \dfrac{\partial}{\partial t_1} s_X(t_1, t_2)$ and

c) $s_{X\dot{X}}(t_1, t_2) = \dfrac{\partial}{\partial t_2} s_X(t_1, t_2)$ are true!

d) How will these equations be changed, if X is stationary!

10.3. A noise voltage across an ideal capacitor C can be approximated by a stationary Gaussian process U with $m_U(t) = 0$ and

$$s_U(\tau) = A^2 \exp\left(-a\tau^2\right) \quad (A \in \mathbb{R},\ a > 0)$$

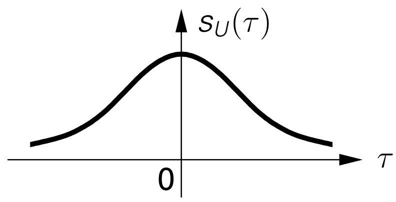


Figure 10.3

(see Figure 10.3).
For the current I through C, calculate

a) the mean m_I $(m_I(t) = ?)$,

b) the cross-correlation functions s_{IU} and s_{UI} $(s_{IU}(\tau) = ?, s_{UI}(\tau) = ?)$ (Draw curves!),

c) the autocorrelation function s_I $(s_I(\tau) = ?)$ (Draw a curve!), and

d) the PDF f_I $(f_I(i, t) = ?)$!

10.4. The circuit (zero state at $t = 0$) given in Figure 10.4 contains a noise voltage source which can be represented by the stationary random process U with the autocorrelation function s_U:

$$s_U(\tau) = 2U_0^2\, \mathrm{e}^{-a|\tau|} \qquad (a > 0).$$

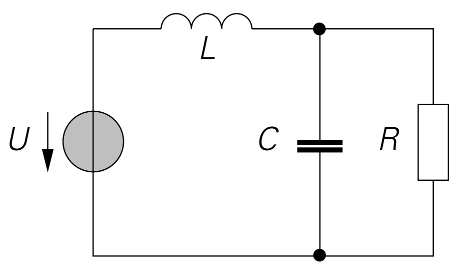


Figure 10.4

Calculate the power density spectra of the voltage U and the current I through R!

10.5. A noise voltage across the terminals of a RLC two terminal network (see Figure 10.5) can be represented by the stationary random process U with a constant power density spectrum $S_U(\omega) = S_0$.

a) Calculate the power density spectrum of the total current I!

b) Calculate the autocorrelation function of the current I_R!

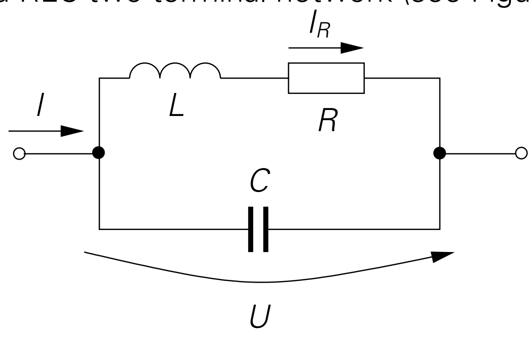


Figure 10.5

10.6. The input of a linear dynamic system with the impulse response g (see Figure 10.6) can be approximated by a stationary random process X with the autocorrelation function s_X.
Calculate the cross-correlation function s_{XY} in integral form as a function of s_X and g! Determine the result specifically for $s_X(\tau) = S_0\,\delta(\tau)$!

Note:

$$Y(t) = \int_0^\infty g(\lambda)X(t-\lambda)\,\mathrm{d}\lambda$$

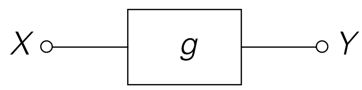


Figure 10.6

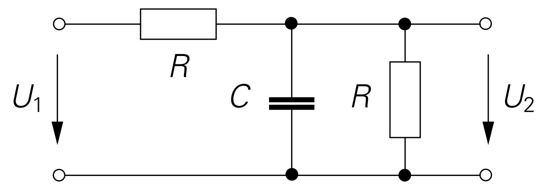
for any stationary random process at arbitrary time t.

10.7. The input voltage X ($\Leftrightarrow U_1$) of a RC network shown in Figure 10.7 can be approximated by a stationary Gaussian process with the power density spectrum $S_X(\omega) = K$ (white noise) and the mean $m_X(t) = 0$.

a) Calculate the power density spectrum $S_Y(\omega)$ of the output voltage Y ($\Leftrightarrow U_2$)!

b) Calculate the autocorrelation function s_Y of the process Y!

c) Specify $f_Y(y, t)$ and $f_Y(y_1, t_1; y_2, t_2)$!



$(X \Leftrightarrow U_1,\ Y \Leftrightarrow U_2)$

Figure 10.7

10.8. In the circuit (zero state at $t = 0$) shown in Figure 10.8, the variable X denotes a stationary random process with a constant power density spectrum $S_X(\omega) = S_0$. Calculate the power density spectrum of the process Y at the output of this system!

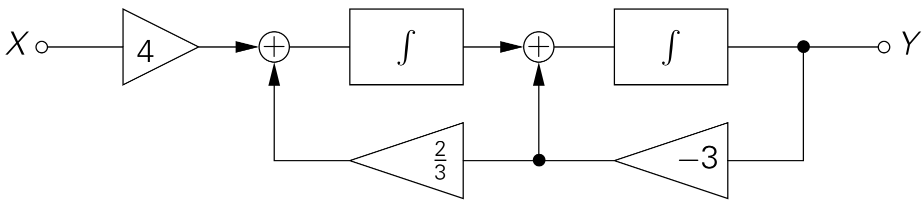


Figure 10.8

10.9. Given an electric network with the thermally noisy ohmic resistors R_1, R_2, and R_3 shown in Figure 10.9, calculate the power density spectrum of the noise voltage U across the terminals AB and draw a noise equivalent circuit!

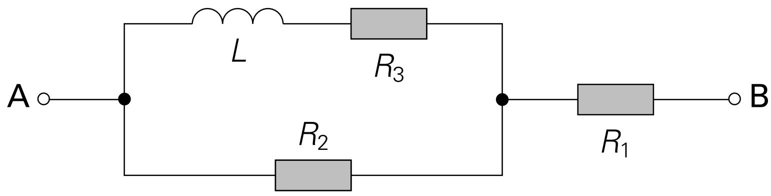


Figure 10.9

10.10. For the quantitative description of a mean-square ergodic random process U, the effective noise voltage

$$U_{\text{eff}} = \sqrt{\overline{u^2(t)}} = \sqrt{\mathrm{E}\left(U^2(t)\right)}.$$

is used.

Calculate the effective noise voltage

a) in case of a given autocorrelation function s_U:

$$s_U(\tau) = A^2\, \mathrm{e}^{-\alpha|\tau|} \cos \beta\tau \qquad (A > 0,\ \alpha > 0,\ \beta > 0);$$

Numerical example: $A = 1$ V, $\alpha = 10^{-3}$ s^{-1}, $\beta = 10^{-4}$ s^{-1};

b) for an ohmic resistor R in a low frequency range, i. e. the power density spectrum S_U is

$$S_U(\omega) = \begin{cases} 2kTR & |\omega| < \omega_0, \\ 0 & |\omega| > \omega_0. \end{cases}$$

Numerical example: $R = 1$ M˙, $f_0 = \frac{\omega_0}{2\pi} = 20$ kHz, $T = 300$ K, $k = 1.38 \cdot 10^{-23}$ Ws/K

10.11. Figure 10.11 shows a block diagram of a circuit for measuring the root mean square (RMS) of weak signals. In this diagram, X denotes the input signal, whose RMS has to be determined, whereas U and V denote the noise signals of the two amplifiers with the amplification factors v_1 and v_2. The given signals X, U, and V can be interpreted as independent stationary and ergodic random process with a zero mean. Show that the output signal is proportional to

$$X_{\text{eff}} = \sqrt{\overline{x^2(t)}} = \sqrt{\mathrm{E}\left(X^2(t)\right)}$$

and independent of the noise voltages of the amplifiers!

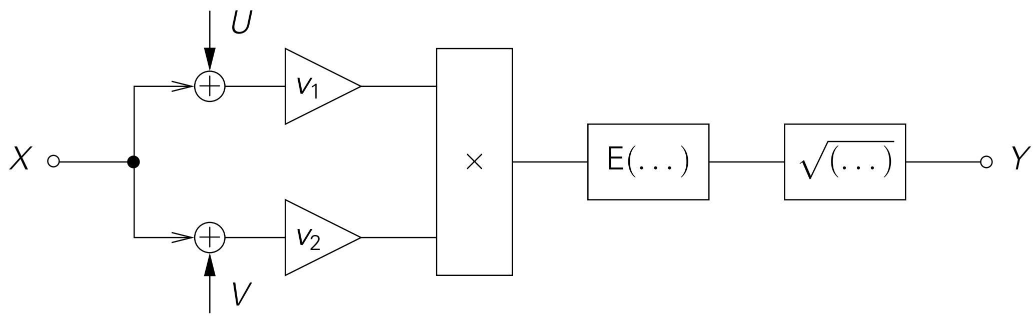


Figure 10.11

11.1. A digital second order band pass filter (see Figure 11.1) is given with the transfer function G:

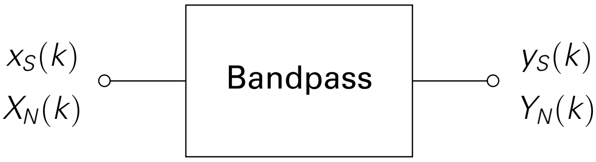


Figure 11.1

$$G(z) = \frac{z^2 - 1}{2.1\,z^2 + 1.9}$$

The input x of the digital filter is the sum of a discrete time signal x_S:

$$x_S(k) = \hat{X} \sin \Omega k \qquad (\hat{X} = 1,\ \Omega = \frac{1}{2}\pi)$$

and a discrete time random signal (caused by a previous analog to digital conversion), which can be approximated by a stationary discrete time random process X_N with uncorrelated signal values ("white noise"). It is assumed, that $X_N(k)$ is uniformly distributed in the intervall $\left(-\frac{1}{2}\Delta, +\frac{1}{2}\Delta\right]$ (numerical example: $\Delta = 2^{-10}$).

a) Calculate the amplitude frequency response of the digital filter and draw a sketch!

b) Calculate the signal-to-noise ratio (SNR) at the input and at the output of the filter! Note:

$$a = 20 \lg \frac{X_{S,\text{eff}}}{X_{N,\text{eff}}} = 10 \lg \frac{\overline{x_S{}^2(k)}}{\overline{x_N{}^2(k)}}$$

EXAMINATIONS

E EXAMINATIONS

Stochastic Signals and Systems
1st final examination

1. By playing dice with two independent dice, a discrete two-dimensional random variable (X_1, X_2) is defined.

a) Calculate the expected values $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$!

b) Calculate the variances $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_2)$!

c) Is it possible to determine the correlation coefficient $\varrho(X_1, X_2)$ without any calculation? (Specify the reasons!)

2. A non-linear static system with one input and one output is given (see Figure 2). The input of the system is the random variable X, which is uniformly distributed in the interval $[0, 3]$. Which characteristic curve φ is required for the system to produce a random variable Y with the probability density function (PDF)

$$f_Y(y) = \begin{cases} 0.5y & 0 \leq y \leq 2 \\ 0 & y < 0, y > 2 \end{cases}$$

at the output?

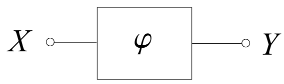


Figure 2: Static system

a) Draw the curves of $f_X(x)$, $F_X(x)$, $f_Y(y)$, and $F_Y(y)$!

b) Calculate $y = \varphi(x)$ and also draw the curve of this system function! In case of multiple results, choose a suitable solution!

3. A noise voltage across an ohmic resistor R can be approximated by a stationary random process U with a zero mean and the power density spectrum S_U:

$$S_U(\omega) = \begin{cases} S_0 & \text{if } -\omega_0 \leq \omega \leq \omega_0 \\ 0 & \text{if } \omega < -\omega_0,\ \omega > \omega_0 \end{cases} \qquad (S_0 > 0,\ \text{constant})$$

a) What is the power density spectrum and the autocorrelation function of the current I through the ohmic resistor R!

b) Calculate the mean power input of R!

4. The circuit shown in Figure 4 contains two adders and two ideal amplifiers with the amplification factors v_1 and v_2.
The processes X (input process), U, and V (disturbance processes) are stationary and independent random processes with the mean values $m_X = m_U = m_V = 0$.
The power density spectrum of the process X is given by

$$S_X(\omega) = \frac{A^2}{\omega^2 + a^2} \qquad (A > 0,\ a > 0)$$

whereas U and V are white noise processes with

$$S_U(\omega) = S_V(\omega) = S_0 \qquad (S_0 > 0).$$

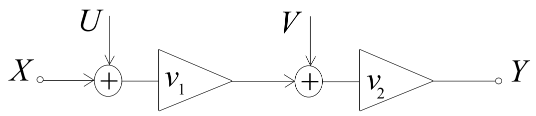


Figure 4: Static system

a) Calculate the cross-correlation function of the processes X and Y!

b) Calculate the power density spectrum of the process Y!

5. In the circuit (zero state at $t = 0$) shown in Figure 5, the variable X denotes a stationary random process with a constant power density spectrum $S_X(\omega) = S_0$. Calculate the power density spectrum of the process Y at the output of this system!

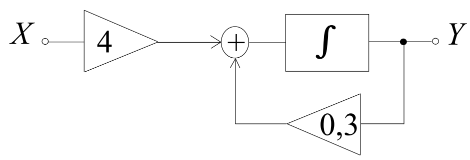


Figure 5: Circuit

6. Figure 6a shows a RLC two terminal network with two thermally noisy ohmic resistors at the same absolute temperature T. Calculate the power density spectra of the noise sources in the noise equivalent circuits given in the Figures 6b and 6c!

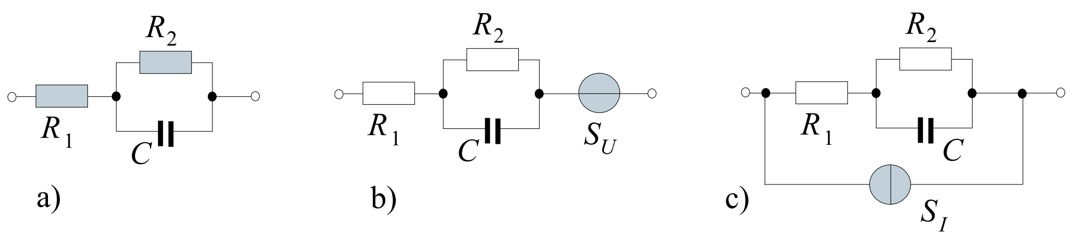


Figure 6: a) RLC two terminal network b) and c) Noise equivalent circuits

Stochastic Signals and Systems
2nd final examination

1. The probability density function (PDF) f_X of the random variable X is given as:

$$f_X(x) = \begin{cases} 0.50 & 0 < x \leq 1 \\ 0.25 & 1 < x \leq 3 \\ 0 & x \leq 0,\ x > 3. \end{cases}$$

a) Draw a sketch of the PDF of this random variable!

b) Calculate the corresponding cumulative distribution function (CDF) and draw a sketch!

c) Calculate the probability $P\{X \geq 2\}$!

d) Calculate the expected value $\mathrm{E}(X)$!

2. The random variables X_1, X_2, and X_3 are stochastically independent of each other with the expected values

$$\mathrm{E}(X_1) = \mathrm{E}(X_2) = \mathrm{E}(X_3) = 0$$

and the variances

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \mathrm{Var}(X_3) = \sigma^2.$$

a) Calculate $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ at the output of the static system shown in Figure 2!

b) Calculate the correlation coefficient $\varrho = \varrho(X_1, Y)$ of the random variables X_1 and Y!

c) How do the results of a) and b) change, if the amplifier V_2 in Figure 2 fails (i. e. $V_2 = 0$)?

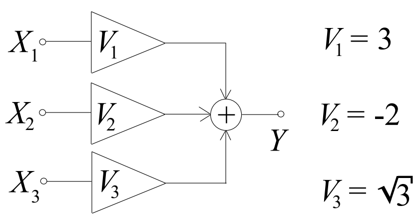


Figure 2 Static system

3. The noise voltage U across the RL series connection shown in Figure 3 has a constant power density spectrum $S_U(\omega) = S_0 = \text{const.}$

a) Calculate the power density spectra of the partial voltages U_L, U_R and of the current I!

b) Calculate the autocorrelation function of the current I!

c) Calculate the mean power input of R!

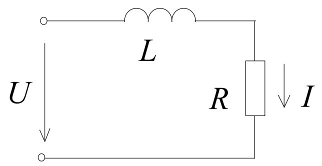


Figure 3 RL series connection

4. Calculate the probability density function (PDF) $f_Y(y, t)$ of the random process

$$Y = aX + b \qquad (a \in \mathbb{R},\ b \in \mathbb{R}),$$

if X is a stationary Gaussian process with a zero mean (i. e. $m_X(t) = 0$) and the autocorrelation function s_X:

$$s_X(\tau) = A^2 \exp(-\alpha|\tau|) \qquad (A > 0,\ \alpha > 0)?$$

5.

a) What is a stationary random processes?

b) Which properties do the expected value and the autocorrelation function of a stationary random process have?

c) Calculate the autocorrelation function of a stationary random process X, if its power density spectrum is given by

$$S_X(\omega) = \begin{cases} S_0 = \text{const} & -\omega_0 \le \omega \le \omega_0 \\ 0 & \omega > \omega_0,\ \omega < -\omega_0 \end{cases} \quad !$$

Draw qualitatively a sketch of the autocorrelation function!

6. Two thermally noisy ohmic resistors R_1 with the absolute temperature T_1 and R_2 with the absolute temperature T_2 are connected in parallel (Figure 6). Calculate the power density spectrum $S_U(\omega)$ of the noise voltages U across the parallel connection!

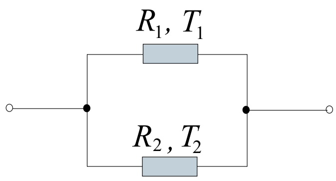


Figure 6 Thermally noisy ohmic resistors

FORMULARY

F FORMULARY

Formulary of *Stochastic Signals and Systems (1)*

Preliminaries of the Probability Calculus

$$P(\overline{A}) = 1 - P(A)$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B), \qquad \text{if } A \cap B = \varnothing \ \ (A,\ B \text{ mutually exclusive}) \end{aligned}$$

$$\begin{aligned} P(A \setminus B) &= P(A) - P(A \cap B) \\ &= P(A) - P(B), \qquad \text{if } B \subset A \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) = P(B|A)P(A) \\ &= P(A)P(B), \qquad \text{if } A \text{ and } B \text{ are stochastically independent} \end{aligned}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0) \qquad \text{conditional probability}$$

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \qquad \text{formula of total probability}$$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} \qquad \text{Bayesian formula}$$

One-dimensional Random Variables

$$F_X(\xi) = P\{X < \xi\} = \int_{-\infty}^{\xi} f_X(x)\ \mathrm{d}x$$

$$P\{a \leq X < b\} = \int_{a}^{b} f_X(x)\ \mathrm{d}x = F_X(b) - F_X(a)$$

Specific Distributions

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \qquad (\sigma > 0) \qquad \text{normal distribution (Gaussian d.)}$$

$$P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k} \qquad (k = 0, 1, 2, \ldots, n) \qquad \text{binomial d. (Bernoulli d.)}$$

$$P\{X = k\} = \frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda} \qquad (k = 0, 1, 2, \ldots) \qquad \text{Poisson distribution}$$

Moments of One-dimensional Random Variables

	X discrete	X continuous
General Moments:		
Expected value $m = \mathrm{E}(X)$	$\sum_i x_i P\{X = x_i\}$	$\int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x$
Moment of n-th Order $m_n = \mathrm{E}(X^n)$	$\sum_i x_i^n P\{X = x_i\}$	$\int_{-\infty}^{\infty} x^n f_X(x)\, \mathrm{d}x$
Central Moments:		
Dispersion, Variance		
$\mu_2 = \mathrm{E}((X-m)^2) = \mathrm{Var}(X)$	$\sum_i (x_i - m)^2 P\{X = x_i\}$	$\int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, \mathrm{d}x$
Central moment of n-th order		
$\mu_n = \mathrm{E}((X-m)^n)$	$\sum_i (x_i - m)^n P\{X = x_i\}$	$\int_{-\infty}^{\infty} (x-m)^n f_X(x)\, \mathrm{d}x$
Characteristic function:		
$\varphi_X(\lambda) = \mathrm{E}\left(\mathrm{e}^{\,\mathrm{j}\lambda X}\right)$	$\sum_i \mathrm{e}^{\,\mathrm{j}\lambda x_i} P\{X = x_i\}$	$\int_{-\infty}^{\infty} \mathrm{e}^{\,\mathrm{j}\lambda x} f_X(x)\, \mathrm{d}x$

Two-dimensional Random Variables $X = (X_1, X_2)$

$$F_X(\xi_1, \xi_2) = P\{X_1 < \xi_1, X_2 < \xi_2\} = \int_{-\infty}^{\xi_1}\int_{-\infty}^{\xi_2} f_X(x_1, x_2)\, \mathrm{d}x_2\, \mathrm{d}x_1$$

$$P\{a_1 \le X_1 < b_1, a_2 \le X_2 < b_2\} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_X(x_1, x_2)\, \mathrm{d}x_2\, \mathrm{d}x_1$$

$$= F_X(b_1, b_2) - F_X(b_1, a_2) - F_X(a_1, b_2) + F_X(a_1, a_2)$$

Marginal probability density function

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_X(x_1, x_2)\, \mathrm{d}x_2 \qquad f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_X(x_1, x_2)\, \mathrm{d}x_1$$

Conditional probability density function

$$f_{X_1}(x_1|x_2) = \frac{f_X(x_1, x_2)}{f_{X_2}(x_2)} \qquad f_{X_2}(x_2|x_1) = \frac{f_X(x_1, x_2)}{f_{X_1}(x_1)}$$

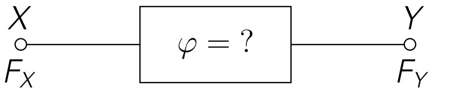
Correlation coefficient

$$\varrho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}} = \frac{\mathrm{E}((X_1 - m_{X_1})(X_2 - m_{X_2}))}{\sqrt{\mathrm{E}\left((X_1 - m_{X_1})^2\right)\mathrm{E}\left((X_2 - m_{X_2})^2\right)}}$$

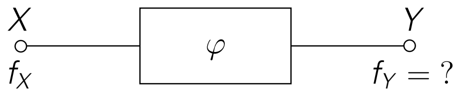
Formulary of *Stochastic Signals and Systems (2)*

Transformation of Random Variables by Static Systems

One-dimensional Random Variables:



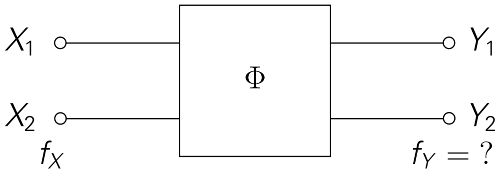
φ bijective, monotonically increasing
$y = \varphi(x) = F_Y^{-1}(F_X(x))$



φ bijective

$$f_Y(y) = \left. \frac{f_X(x)}{\left|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|} \right|_{x=\varphi^{-1}(y)}$$

Two-dimensional Random Variables:



Φ bijective

$$f_Y(y_1, y_2) = \left. \frac{f_X(x_1, x_2)}{\left|\frac{\partial(\varphi_1,\varphi_2)}{\partial(x_1,x_2)}\right|} \right|_{(x_1,x_2)=\Phi^{-1}(y_1,y_2)}$$

Random Processes

Expected value

$$m_X(t) = \mathrm{E}(X(t)) = \int_{-\infty}^{\infty} x f_X(x, t)\ \mathrm{d}x$$

Variance

$$\mathrm{Var}(X(t)) = \mathrm{E}\left(((X(t) - m_X(t))^2\right) = \int_{-\infty}^{\infty} (x - m_X(t))^2 f_X(x, t)\ \mathrm{d}x$$

Autocorrelation function

$$s_X(t_1, t_2) = \mathrm{E}(X(t_1)X(t_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, t_1; x_2, t_2)\ \mathrm{d}x_1\ \mathrm{d}x_2$$

Cross-correlation function

$$s_{XY}(t_1, t_2) = \mathrm{E}(X(t_1)Y(t_2)) = s_{YX}(t_2, t_1)$$

Covariance function

$$\begin{aligned}\mathrm{Cov}(X(t_1), X(t_2)) &= \mathrm{E}((X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))) \\ &= s_X(t_1, t_2) - m_X(t_1)m_X(t_2)\end{aligned}$$

Covariance matrix

$$\mathrm{Cov}(X) = \begin{pmatrix} \mathrm{Cov}(X(t_1), X(t_1)) & \cdots & \mathrm{Cov}(X(t_1), X(t_n)) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(X(t_n), X(t_1)) & \cdots & \mathrm{Cov}(X(t_n), X(t_n)) \end{pmatrix}$$

Stationary Random Processes

Expected value	$\mathrm{E}(X(t)) = m_X(t) = m_X$ (= const.)
Variance	$\mathrm{Var}(X(t)) = {\sigma_X}^2$ (= const.)
Autocorrelation function	$s_X(\tau) = \mathrm{E}(X(t)X(t+\tau))$
Cross-correlation function	$s_{XY}(\tau) = \mathrm{E}(X(t)Y(t+\tau)) = s_{YX}(-\tau)$
Power density spectrum	$S_X(\omega) = \int\limits_{-\infty}^{\infty} s_X(\tau)\, \mathrm{e}^{-\mathrm{j}\omega\tau}\, \mathrm{d}\tau$
(Theorem of Wiener/Chintschin)	$s_X(\tau) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} S_X(\omega)\, \mathrm{e}^{\mathrm{j}\omega\tau}\, \mathrm{d}\omega$

Gaussian Processes

$$f_X(x_1, t_1; \ldots; x_n, t_n) = \frac{1}{\sqrt{(2\pi)^n \det C}} \exp\left(-\frac{1}{2}(x-m)C^{-1}(x-m)'\right)$$

With:

$(x-m) = (x_1 - m_X(t_1) \;\cdots\; x_n - m_X(t_n))$	row matrix	
$(x-m)'$	transposed matrix of $(x-m)$	
$C = \mathrm{Cov}(X)$	covariance matrix with the elements $\mathrm{Cov}(X(t_i), X(t_j)) = s_X(t_i, t_j) - m_X(t_i)m_X(t_j)$	

Markov Processes

$$f_X(x_n, t_n | x_1, t_1; \ldots; x_{n-1}, t_{n-1}) = f_X(x_n, t_n | x_{n-1}, t_{n-1}) \qquad (t_1 < t_2 < \cdots < t_n)$$

$$f_X(x_1, t_1; \ldots; x_n, t_n) = f_X(x_n, t_n | x_{n-1}, t_{n-1}) \cdot f_X(x_{n-1}, t_{n-1} | x_{n-2}, t_{n-2}) \cdots f_X(x_2, t_2 | x_1, t_1) \cdot f_X(x_1, t_1)$$

$$f_X(x_1, t_1; \ldots; x_n, t_n) = \frac{f_X(x_n, t_n; x_{n-1}, t_{n-1})}{f_X(x_{n-1}, t_{n-1})} \cdots \frac{f_X(x_2, t_2; x_1, t_1)}{f_X(x_1, t_1)} \cdot f_X(x_1, t_1)$$

Formulary of *Stochastic Signals and Systems (3)*

Analysis of Random Processes

Mean-square convergence of the sequence $X = (X_i)_{i\in\mathbb{N}}$ of random variables:

$$\underset{i\to\infty}{\mathrm{l.i.m.}}\, X_i = X \qquad \lim_{i\to\infty} ||X_i - X|| = 0 \qquad \mathrm{E}\left(\underset{i\to\infty}{\mathrm{l.i.m.}}\, X_i\right) = \lim_{i\to\infty} \mathrm{E}(X_i)$$

Mean-square continuity of a random process $X = (X_t)_{t\in\mathbb{T}}$:

$$\underset{\tau\to 0}{\mathrm{l.i.m.}}\, X(t+\tau) = X(t) \qquad \lim_{\tau\to 0} ||X(t+\tau) - X(t)|| = 0$$

Mean-square differentiation of a random process $X = (X_t)_{t\in\mathbb{T}}$:

$$\dot{X}(t) = \underset{\tau\to 0}{\mathrm{l.i.m.}}\, \frac{X(t+\tau) - X(t)}{\tau}$$

For mean-square differentiable random processes $X = (X_t)_{t\in\mathbb{T}}$ are:

Expected value $\quad m_{\dot{X}}(t) = \mathrm{E}\left(\dot{X}(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t} m_X(t)$

Autocorrelation function $\quad s_{\dot{X}}(t_1, t_2) = \mathrm{E}\left(\dot{X}(t_1)\dot{X}(t_2)\right) = \frac{\partial^2}{\partial t_1\, \partial t_2} s_X(t_1, t_2)$

Cross-correlation function $\quad s_{\dot{X}X}(t_1, t_2) = \mathrm{E}\left(\dot{X}(t_1)X(t_2)\right) = \frac{\partial}{\partial t_1} s_X(t_1, t_2)$

$\quad s_{X\dot{X}}(t_1, t_2) = \mathrm{E}\left(X(t_1)\dot{X}(t_2)\right) = \frac{\partial}{\partial t_2} s_X(t_1, t_2)$

For mean-square differentiable stationary random processes $X = (X_t)_{t\in\mathbb{T}}$ are:

Expected value $\quad m_{\dot{X}}(t) = \mathrm{E}\left(\dot{X}(t)\right) = 0$

Autocorrelation function $\quad s_{\dot{X}}(\tau) = \mathrm{E}\left(\dot{X}(t)\dot{X}(t+\tau)\right) = -\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} s_X(\tau)$

Cross-correlation function $\quad s_{\dot{X}X}(\tau) = \mathrm{E}\left(\dot{X}(t)X(t+\tau)\right) = -\frac{\mathrm{d}}{\mathrm{d}\tau} s_X(\tau)$

$\quad s_{X\dot{X}}(\tau) = \mathrm{E}\left(X(t)\dot{X}(t+\tau)\right) = \frac{\mathrm{d}}{\mathrm{d}\tau} s_X(\tau)$

Let $X = (X_t)_{t\in\mathbb{T}}$ be a mean-square integrable random process and f a deterministic function, then

$$\mathrm{E}\left(\int_a^b f(t,\tau)X(t)\,\mathrm{d}t\right) = \int_a^b f(t,\tau)\,\mathrm{E}(X(t))\,\mathrm{d}t$$

hold.

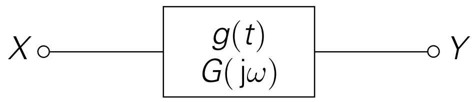
Ergodic Random Processes

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,\mathrm{d}t = \mathrm{E}(X(t)) = m_X(t) = m_X = \text{const.}$$

$$\overline{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\,\mathrm{d}t = \mathrm{E}(X(t)X(t+\tau)) = s_X(\tau)$$

Linear Dynamic Systems

Given: stationary random process X



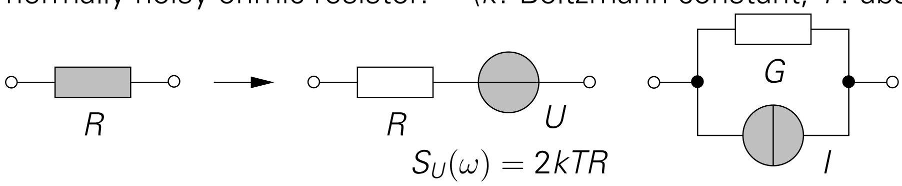
System's output process:	$Y(t) = \int_{-\infty}^{t} g(t-\tau)X(\tau)\,\mathrm{d}\tau = \int_{0}^{\infty} g(\tau)X(t-\tau)\,\mathrm{d}\tau$
Expected value	$m_Y(t) = m_X \int_{0}^{\infty} g(\tau)\,\mathrm{d}\tau \quad (m_X(t) = m_X = \text{const.})$
Autocorrelation function	$s_Y(\tau) = \int_0^\infty\int_0^\infty g(\tau_1)g(\tau_2)s_X(\tau+\tau_1-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$
Cross-correlation function	$s_{XY}(\tau) = \int_0^\infty g(\tau_1)s_X(\tau-\tau_1)\,\mathrm{d}\tau_1$
Power density spectrum	$S_Y(\omega) = \lvert G(\,\mathrm{j}\omega)\rvert^2 S_X(\omega)$
Cross-power density spectrum	$S_{XY}(\omega) = G(\,\mathrm{j}\omega)S_X(\omega)$

Calculation of the cross-correlation function at the system output by the residual method:

$$s_Y(\tau) = \sum_{\mathrm{Re}(s)<0} \mathrm{Res}\ G(s)G(-s)\left(\tilde{S}_X(s) + \tilde{S}_X(-s)\right)\mathrm{e}^{s|\tau|} \qquad \text{with}$$

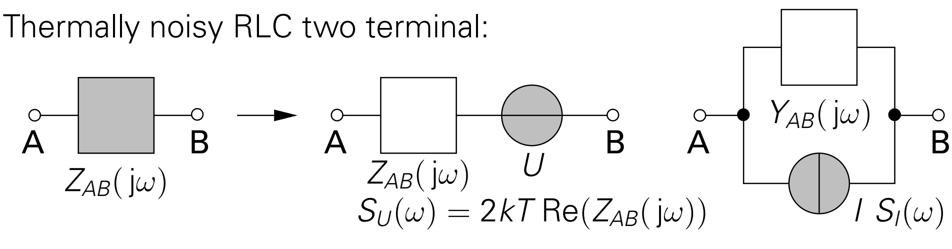
$$\tilde{S}_X(s) = \int_0^\infty \tilde{s}_X(\tau)\,\mathrm{e}^{-s\tau}\,\mathrm{d}\tau \qquad \tilde{s}_X(\tau) = \begin{cases} s_X(\tau) & \tau \ge 0, \\ 0 & \tau < 0. \end{cases}$$

Thermally noisy ohmic resistor: (k: Boltzmann constant, T: absolute temperature)



$$G = \frac{1}{R} \qquad S_I(\omega) = 2kTG$$

Thermally noisy RLC two terminal:



$$Y_{AB}(\,\mathrm{j}\omega) = \frac{1}{Z_{AB}(\,\mathrm{j}\omega)}$$

$$S_I(\omega) = 2kT\,\mathrm{Re}(Y_{AB}(\,\mathrm{j}\omega))$$

FORMULARY OF LTI-SYSTEMS

G FORMULARY OF LTI-SYSTEMS

Formulary of *Analog Signals and Systems (1)*

Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

Rules of the Fourier Transform:

No.	$x(t)$	$X(\omega)$	Remark
1	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$	Linearity
2	$x(t-\tau)$	$\mathrm{e}^{-\mathrm{j}\omega\tau} X(\omega)$	Displacement law (time shift)
3	$x(t)\,\mathrm{e}^{\mathrm{j}\omega_0 t}$	$X(\omega-\omega_0)$	Displacement law (frequency shift)
4	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$	Similarity law $(a \neq 0)$
5	$\dot{x}(t)$	$\mathrm{j}\omega X(\omega)$	Differentiation law
6	$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{\mathrm{j}\omega} X(\omega)$	Integration law *)
7	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,\mathrm{d}\tau$	$X_1(\omega)X_2(\omega)$	Convolution law (time domain)
8	$x_1(t)x_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(u)X_2(\omega-u)\,\mathrm{d}u$	Convolution law (frequency domain)
9	If the correspondence $x(t) \circ\!\!-\!\!\bullet\ X(\omega)$ is true, then the correspondence $X(t) \circ\!\!-\!\!\bullet\ 2\pi x(-\omega)$ is true too.		Exchange law

*) It has to be proven that the Fourier transform of the integral on the left-hand side really exists!

Correspondences of the Fourier Transform:

No.	$x(t)$	$X(\omega)$	
1	$\delta(t)$	1	
2	$\mathbf{1}(t)$	$\pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}$	
3	$\mathrm{Rect}\left(\frac{t}{2\tau}\right) = \begin{cases} 1 & -\tau \le t \le \tau \\ 0 & t < -\tau \vee t > \tau \end{cases}$	$2\tau\frac{\sin(\omega\tau)}{\omega\tau} = 2\tau\,\mathrm{si}(\omega\tau)$	
4	$\frac{\omega_0}{\pi}\,\mathrm{si}(\omega_0 t) = \frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t}$	$\mathrm{Rect}\left(\frac{\omega}{2\omega_0}\right) = \begin{cases} 1 & -\omega_0 \le \omega \le \omega_0 \\ 0 & \omega < -\omega_0 \vee \omega > \omega_0 \end{cases}$	$(\omega_0 \neq 0)$
5	$\begin{cases} \mathrm{e}^{-at} & t > 0 \\ 0 & t < 0 \end{cases}$	$\frac{1}{\mathrm{j}\omega + a}$	$(a > 0)$
6	$\mathrm{e}^{-a\|t\|}$	$\frac{2a}{\omega^2 + a^2}$	$(a > 0)$
7	$\frac{1}{t^2 + a^2}$	$\frac{\pi}{a}\,\mathrm{e}^{-a\|\omega\|}$	$(a > 0)$
8	e^{-at^2}	$\sqrt{\frac{\pi}{a}}\;\mathrm{e}^{-\frac{\omega^2}{4a}}$	$(a > 0)$
9	$(1 + a\|t\|)\,\mathrm{e}^{-a\|t\|}$	$\frac{4a^3}{(\omega^2 + a^2)^2}$	$(a > 0)$
10	$\left(1 + a\|t\| + \frac{1}{3}(at)^2\right)\mathrm{e}^{-a\|t\|}$	$\frac{16a^5}{3(\omega^2 + a^2)^3}$	$(a > 0)$
11	$\mathrm{e}^{-a\|t\|}\cos(\beta t)$	$\frac{2a(\omega^2 + a^2 + \beta^2)}{(\omega^2 - a^2 - \beta^2)^2 + 4a^2\omega^2}$	$(a > 0)$
12	$\mathrm{e}^{-a\|t\|}\left(\cos(\beta t) + \frac{a}{\beta}\sin(\beta\|t\|)\right)$	$\frac{4a(a^2 + \beta^2)}{((\omega - \beta)^2 + a^2)((\omega + \beta)^2 + a^2)}$	$(a > 0)$
13	$\begin{cases} a\left(1 - \frac{\|t\|}{\tau}\right) & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$	$\frac{4a}{\omega^2\tau}\sin^2\left(\frac{\omega\tau}{2}\right) = a\tau\,\mathrm{si}^2\left(\frac{\omega\tau}{2}\right)$	$(\tau \neq 0)$
14	$\cos(\omega_0 t)$	$\pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$	
15	$\sin(\omega_0 t)$	$\mathrm{j}\pi\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$	

Formulary of *Analog Signals and Systems (2)*

Laplace Transform:

$$X(s) = \int_0^{\infty} x(t)\,\mathrm{e}^{-st}\;\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi\,\mathrm{j}} \int_{\delta-\mathrm{j}\infty}^{\delta+\mathrm{j}\infty} X(s)\,\mathrm{e}^{st}\;\mathrm{d}s$$

Rules of the Laplace Transform:

No.	$x(t)$	$X(s)$	Remark
1	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$	Linearity
2	$x(t-\tau) \quad (\tau > 0)$	$\mathrm{e}^{-s\tau} X(s)$	Displacement law
3	$x(at)$	$\frac{1}{a} X\left(\frac{s}{a}\right)$	Similarity law $(a > 0)$
4	$\dot{x}(t)$	$sX(s) - x(+0)$	Differentiation law
5	$\int_0^t x(\tau)\;\mathrm{d}\tau$	$\frac{1}{s} X(s)$	Integration law
6	$\mathrm{e}^{-at} x(t)$	$X(s+a)$	Attenuation law
7	$\int_0^t x_1(\tau) x_2(t-\tau)\;\mathrm{d}\tau$	$X_1(s) X_2(s)$	Convolution law
8	$x(t) = \sum_i \operatorname*{Res}_{s=s_i} \left[X(s)\,\mathrm{e}^{st}\right]$		Residual formula,

where

$$\operatorname*{Res}_{s=s_i} \left[X(s)\,\mathrm{e}^{st}\right] = \frac{1}{(m-1)!} \lim_{s \to s_i} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left[X(s)\,\mathrm{e}^{st} (s-s_i)^m\right]$$

with s_i: m-fold pole of $X(s)$,

and $X(s)$ rational with $X(\infty) \to 0$.

Correspondences of the Laplace Transform:

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$\mathbf{1}(t)$	$\frac{1}{s}$
3	$t\ \mathbf{1}(t)$	$\frac{1}{s^2}$
4	$e^{at}\ \mathbf{1}(t)$	$\frac{1}{s-a}$
5	$t\,e^{at}\ \mathbf{1}(t)$	$\frac{1}{(s-a)^2}$
6	$\frac{t^{n-1}}{(n-1)!}e^{at}\ \mathbf{1}(t)$	$\frac{1}{(s-a)^n} \qquad (n=1,2,3,\dots)$
7	$\cos at\ \mathbf{1}(t)$	$\frac{s}{s^2+a^2}$
8	$\sin at\ \mathbf{1}(t)$	$\frac{a}{s^2+a^2}$
9	$\cosh at\ \mathbf{1}(t)$	$\frac{s}{s^2-a^2}$
10	$\sinh at\ \mathbf{1}(t)$	$\frac{a}{s^2-a^2}$
11	$e^{at}\cos\beta t\ \mathbf{1}(t)$	$\frac{s-a}{(s-a)^2+\beta^2}$
12	$e^{at}\sin\beta t\ \mathbf{1}(t)$	$\frac{\beta}{(s-a)^2+\beta^2}$
13	$e^{at}\left(\cos\beta t+\frac{a}{\beta}\sin\beta t\right)\ \mathbf{1}(t)$	$\frac{s}{(s-a)^2+\beta^2}$
14	$\cos^2 at\ \mathbf{1}(t)$	$\frac{s^2+2a^2}{s(s^2+4a^2)}$
15	$\sin^2 at\ \mathbf{1}(t)$	$\frac{2a^2}{s(s^2+4a^2)}$
16	$\cos(at+b)\ \mathbf{1}(t)$	$\frac{s\cos b-a\sin b}{s^2+a^2}$
17	$\sin(at+b)\ \mathbf{1}(t)$	$\frac{s\sin b+a\cos b}{s^2+a^2}$
18	$\frac{1}{\sqrt{\pi t}}\ \mathbf{1}(t)$	$\frac{1}{\sqrt{s}}$
19	$2\sqrt{\frac{t}{\pi}}\ \mathbf{1}(t)$	$\frac{1}{s\sqrt{s}}$

Formulary of *Analog Signals and Systems (3)*

Z-Transform:

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$x(k) = \frac{1}{2\pi \mathrm{j}} \oint_c X(z) z^{k-1} \, \mathrm{d}z \qquad (k = 0, 1, 2, \ldots)$$

Rules of the Z-Transform:

No.	$x(k)$	$X(z)$	Remark
1	$\alpha x_1(k) + \beta x_2(k)$	$\alpha X_1(z) + \beta X_2(z)$	Linearity
2	$x(k-m)$	$z^{-m} X(z)$	Displacement law (→)
3	$x(k+m)$	$z^m \left(X(z) - \sum_{i=0}^{m-1} x(i) z^{-i} \right)$	Displacement law (←)
4	$x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$	Forward difference
5	$x(k) - x(k-1)$	$\left(1 - z^{-1}\right) X(z)$	Backward difference
6	$\sum_{i=0}^{k} x_1(i) x_2(k-i)$	$X_1(z) X_2(z)$	Convolution law
7	$a^k x(k)$	$X\left(\frac{z}{a}\right)$	Attenuation law
8	$\sum_{i=0}^{k} x(i)$	$\frac{z}{z-1} X(z)$	Summation
9	$kx(k)$	$-z \frac{\mathrm{d}}{\mathrm{d}z} X(z)$	Differentiation law (frequency domain)
10	$\frac{1}{k} x(k)$	$\int_z^{\infty} X(w) \frac{\mathrm{d}w}{w}$	Integration (frequency domain)
11	$x(k) = \sum_i \operatorname*{Res}_{z=z_i} \left[X(z) z^{k-1}\right]$		Residual formula,

where

$$\operatorname*{Res}_{z=z_i} \left[X(z) z^{k-1}\right] = \frac{1}{(m-1)!} \lim_{z \to z_i} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[X(z) z^{k-1} (z - z_i)^m\right]$$

with z_i: m-fold pole of $X(z) z^{k-1}$.

Correspondences of the Z-Transform:

No.	$x(k)$	$X(z)$
1	$\delta(k)$	1
2	$\mathbf{1}(k)$	$\dfrac{z}{z-1}$
3	$k\ \mathbf{1}(k)$	$\dfrac{z}{(z-1)^2}$
4	$k^2\ \mathbf{1}(k)$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$a^k\ \mathbf{1}(k)$	$\dfrac{z}{z-a}$
6	$ka^k\ \mathbf{1}(k)$	$\dfrac{az}{(z-a)^2}$
7	$k^2a^k\ \mathbf{1}(k)$	$\dfrac{az(z+a)}{(z-a)^3}$
8	$\dfrac{a^k}{k!}\ \mathbf{1}(k)$	$\mathrm{e}^{\frac{a}{z}}$
9	$\dbinom{k}{m}a^k\ \mathbf{1}(k)$	$\dfrac{a^m z}{(z-a)^{m+1}}$
10	$\mathrm{e}^{ak}\ \mathbf{1}(k)$	$\dfrac{z}{z-\mathrm{e}^{a}}$
11	$k\,\mathrm{e}^{ak}\ \mathbf{1}(k)$	$\dfrac{\mathrm{e}^{a}z}{(z-\mathrm{e}^{a})^2}$
12	$a^k \sin\Omega k\ \mathbf{1}(k)$	$\dfrac{az\sin\Omega}{z^2-2az\cos\Omega+a^2}$
13	$a^k \cos\Omega k\ \mathbf{1}(k)$	$\dfrac{z(z-a\cos\Omega)}{z^2-2az\cos\Omega+a^2}$
14	$a^k \sinh\beta k\ \mathbf{1}(k)$	$\dfrac{az\sinh\beta}{z^2-2az\cosh\beta+a^2}$
15	$a^k \cosh\beta k\ \mathbf{1}(k)$	$\dfrac{z(z-a\cosh\beta)}{z^2-2az\cosh\beta+a^2}$
16	$(-1)^k\ \mathbf{1}(k)$	$\dfrac{z}{z+1}$

Formulary of *Analog Signals and Systems (4)*

Survey of Integral Transforms of Spectral Analysis

	Continuous-time Signal	Discrete-time Signal
Periodic resp. Periodically Continued Signal	**FOURIER-Series** $X_n = \frac{1}{T}\int_T x(t)\ \mathrm{e}^{-\mathrm{j}n\omega_0 t}\ \mathrm{d}t$ $x(t) = \sum_{n=-\infty}^{\infty} X_n\,\mathrm{e}^{\mathrm{j}n\omega_0 t}$	**DFT (FFT)** $X(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\ \mathrm{e}^{-\mathrm{j}2\pi\frac{nk}{N}}$ $x(k) = \sum_{n=0}^{N-1} X(n)\ \mathrm{e}^{\mathrm{j}2\pi\frac{kn}{N}}$
Non-periodic Signal	**FOURIER-Transform** $X(\omega) = \int_{-\infty}^{\infty} x(t)\ \mathrm{e}^{-\mathrm{j}\omega t}\ \mathrm{d}t$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\ \mathrm{e}^{\mathrm{j}\omega t}\ \mathrm{d}\omega$	**DTFT / z-Transform** $X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{k=-\infty}^{\infty} x(k)\ \mathrm{e}^{-\mathrm{j}\omega k\Delta t}$ $\Longrightarrow \sum_{k=-\infty}^{\infty} x(k)\,z^{-k} = X(z)$ $x(k) = \frac{\Delta t}{2\pi}\int_{-\frac{\pi}{\Delta t}}^{\frac{\pi}{\Delta t}} X(\mathrm{e}^{\mathrm{j}\omega})\ \mathrm{e}^{\mathrm{j}\omega k\Delta t}\ \mathrm{d}\omega$ $= \frac{1}{2\pi j}\oint X(z)\,z^{k-1}\ \mathrm{d}z$

Sampling $x(k) = x(t)|_{t=k\cdot\Delta t}$

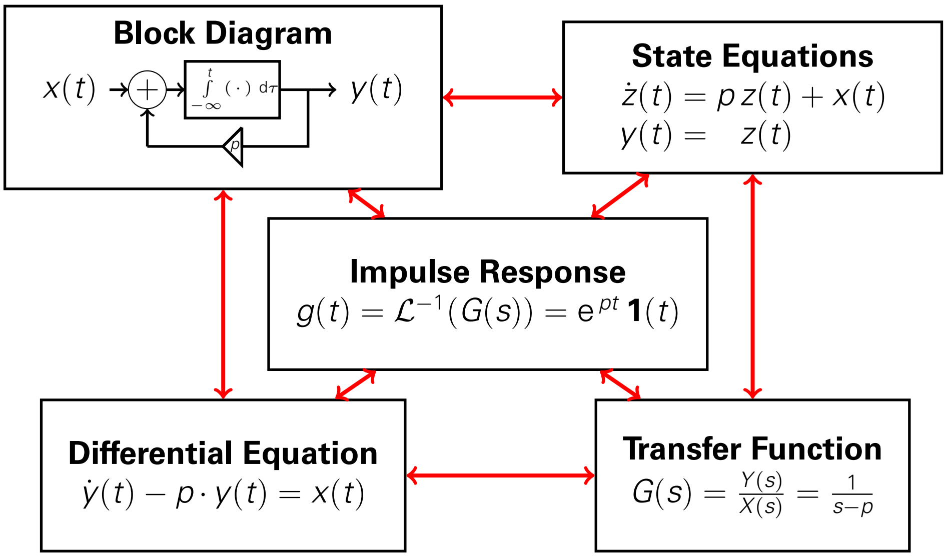
Reconstruction (Sampling Series) $x(t) = \sum_{k=-\infty}^{\infty} x(k)\,\mathrm{si}\left(\frac{\pi}{\Delta t}(t-k\Delta t)\right)$

Energy E of the Energy Signal x

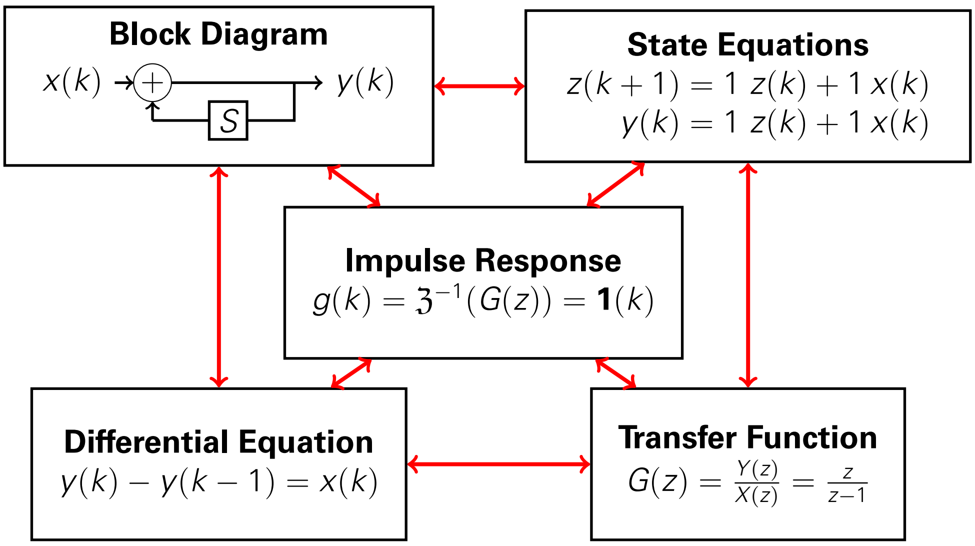
continuous-time: $\int_{-\infty}^{\infty} x^2(t)\ \mathrm{d}t$

discrete-time: $\sum_{-\infty}^{\infty} \Delta t\, x^2(k)$

Presentation of Linear Continuous-time Systems
by means of a simple example



Presentation of Linear Discrete-time Systems
by means of a simple example

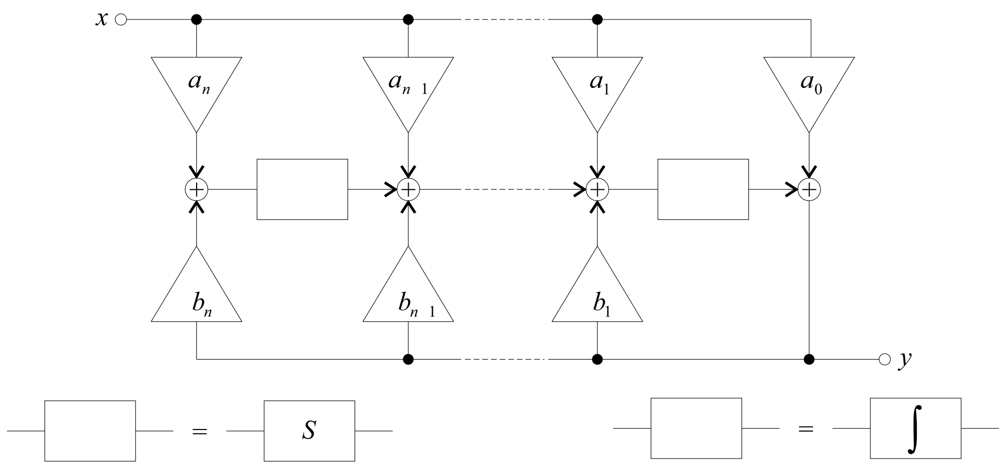


Formulary of *Analog Signals and Systems (5)*

Linear Time Invariant Systems with	
Discrete Time	Continuous Time
State equations:	
$z(k+1) = Az(k) + Bx(k)$	$\dot{z}(t) = Az(t) + Bx(t)$
$y(k) = Cz(k) + Dx(k)$	$y(t) = Cz(t) + Dx(t)$
Fundamental matrix in the frequency domain:	
$\Phi(z) = (zE - A)^{-1} z$	$\Phi(s) = (sE - A)^{-1}$
Transfer matrix resp. transfer function:	
$G(z) = C(zE - A)^{-1} B + D$	$G(s) = C(sE - A)^{-1} B + D$
Solution of the 1st state equation in the frequency domain:	
$Z(z) = \Phi(z)z(0) + \Phi(z)z^{-1}BX(z)$	$Z(s) = \Phi(s)z(0) + \Phi(s)BX(s)$
Input-output-equation in the frequency domain:	
$Y(z) = C\Phi(z)z(0) + G(z)X(z)$	$Y(s) = C\Phi(s)z(0) + G(s)X(s)$
Fundamental matrix (fundamental solution) in the time domain:	
$\varphi(k) = A^k$	$\varphi(t) = \mathrm{e}^{At} = E + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \dots$
Impulse response:	
$g(k) = \begin{cases} D & k = 0 \\ C\varphi(k-1)B & k = 1, 2, \dots \end{cases}$	$g(t) = C\varphi(t)B + D\delta(t)$
Solution of the 1st state equation in the time domain:	
$z(k) = \varphi(k)z(0) + \sum_{i=0}^{k-1} \varphi(k-i-1)Bx(i)$	$z(t) = \varphi(t)z(0) + \int_0^t \varphi(t-\tau)Bx(\tau)\,\mathrm{d}\tau$
Input-output-equation in the time domain:	
$y(k) = C\varphi(k)z(0) + \sum_{i=0}^{k} g(k-i)x(i)$	$y(t) = C\varphi(t)z(0) + \int_0^t g(t-\tau)x(\tau)\,\mathrm{d}\tau$

Linear Time Invariant Systems with

Discrete Time	Continuous Time
Amplitude frequency response (magnitude response):	
$A(\Omega) = \left\lvert G\left(\mathrm{e}^{\,\mathrm{j}\Omega}\right)\right\rvert = \sqrt{G(z)G(z^{-1})}\Big\rvert_{z=\mathrm{e}^{\,\mathrm{j}\Omega}}$	$A(\omega) = \lvert G(\,\mathrm{j}\omega)\rvert = \sqrt{G(s)G(-s)}\Big\rvert_{s=\mathrm{j}\omega}$
Phase frequency response:	
$\varphi(\Omega) = \arg G\left(\mathrm{e}^{\,\mathrm{j}\Omega}\right)$	$\varphi(\omega) = \arg G(\,\mathrm{j}\omega)$
Attenuation:	
$a(\Omega) = -\ln A(\Omega)$ in Np $a(\Omega) = -20\lg A(\Omega)$ in dB	$a(\omega) = -\ln A(\omega)$ in Np $a(\omega) = -20\lg A(\omega)$ in dB
Phase:	
$b(\Omega) = -\arg G\left(\mathrm{e}^{\,\mathrm{j}\Omega}\right)$	$b(\omega) = -\arg G(\,\mathrm{j}\omega)$
Canonical Realisation:	
$G(z) = \dfrac{a_n z^{-n} + \ldots + a_2 z^{-2} + a_1 z^{-1} + a_0}{b_n z^{-n} + \ldots + b_2 z^{-2} + b_1 z^{-1} + 1}$	$G(s) = \dfrac{a_n s^{-n} + \ldots + a_2 s^{-2} + a_1 s^{-1} + a_0}{b_n s^{-n} + \ldots + b_2 s^{-2} + b_1 s^{-1} + 1}$



Discrete Time	Continuous Time
Difference equation:	**Differential equation:**
$y(k+n) + b_1 y(k+n-1) + \ldots + b_n y(k)$ $= a_0 x(k+n) + a_1 x(k+n-1)$ $+ \ldots + a_n x(k)$	$y^{(n)}(t) + b_1 y^{(n-1)}(t) + \ldots + b_n y(t)$ $= a_0 x^{(n)}(t) + a_1 x^{(n-1)}(t)$ $+ \ldots + a_n x(t)$

$$(b_0 = 1, \quad a_i \in \mathbb{R}, \quad b_j \in \mathbb{R})$$