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2. Exercise on Convex Optimization

Problem 5: (Characterization of Differentiable Convex Functions)

Let C be a convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Show that:

a) f is convex over C if and only if

$$f(z) \geq f(x) + (z - x)' \nabla f(x), \quad x, z \in C.$$

b) f is strictly convex over C if and only if the above inequality is strict whenever $x \neq z$.

Problem 6: (Characterization of Twice Continuously Differentiable Convex Functions)

Let C be a convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Show that:

a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex over C.

b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.

c) If C is open and f is convex over C, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Problem 7: (Characterization of Differentiable Convex Functions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Show that f is convex over a nonempty convex set C if and only if

$$\big[\nabla f(x) - \nabla f(y)\big]'(x - y) \geq 0, \quad x, y \in C.$$

Note: The condition above says that the function f, restricted to the line segment connecting x and y, has monotonically nondecreasing gradient.