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1. Exercise on Convex Optimization

Problem 1: (Properties of Convex Sets)

Show that:

a) The intersection $\cap_{i\in I} C_i$ of any collection $\{C_i \,|\, i \in I\}$ of convex sets is convex.

b) The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex.

c) The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex set and λ_1, λ_2 are positive scalars,

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

Show by example that this need not be true when C is not convex.

d) The closure and the interior of a convex set are convex.

e) The image and the inverse image of a convex set under an affine function are convex.

Problem 2: (Properties of Cones)

Show that:

a) The intersection $\cap_{i\in I} C_i$ of a collection $\{C_i \,|\, i \in I\}$ of cones is a cone.

b) The Cartesian product $C_1 \times C_2$ of two cones C_1 and C_2 is a cone.

c) The vector sum $C_1 + C_2$ of two cones C_1 and C_2 is a cone.

d) The closure of a cone is a cone.

e) The image and the inverse image of a cone under a linear transformation is a cone.

Problem 3: (Convexity under Composition)

Let $C \subset \mathbb{R}^n$ nonempty convex. Let $f = (f_1, \ldots, f_m)$, where each $f_i : C \to \mathbb{R}$ is a convex function, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a function that is convex and monotonically non-decreasing over a convex set $D \supset \{f(x) \,|\, x \in C\}$. Show that the function h defined by $h(x) = g[f(x)]$ is convex over $C \times \cdots \times C$.

Problem 4: (Examples of Convex and Concave Functions)

Show that the following functions from $\mathbb{R}^n$ to $(-\infty, \infty]$ are convex:

a) $f_1(x) := \|x\|^p$ with $p \geq 1$.

b) $f_2(x) := 1/f(x)$, where f is concave and $f(x)$ is a positive number for all x.

c) $f_3(x) := \alpha f(x) + \beta$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and α, β are scalars such that $\alpha \geq 0$.

d) $f_4(x) := e^{\beta x' A x}$, where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite and $\beta > 0$.

e) $f_5(x) := \|Ax - b\|_2^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

f) $f_6^*(y) := \sup_{x\in \mathrm{dom}\, f}[y'x - f(x)]$, where $f : \mathbb{R}^n \to \mathbb{R}$.

Moreover show that the following function from $\mathbb{R}^{n\times n}$ to $(-\infty, \infty]$ is concave:

g) $f_7(X) := \log\det(X)$, where $X \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.